| Exercises | DERIVATIVE | Y11 |
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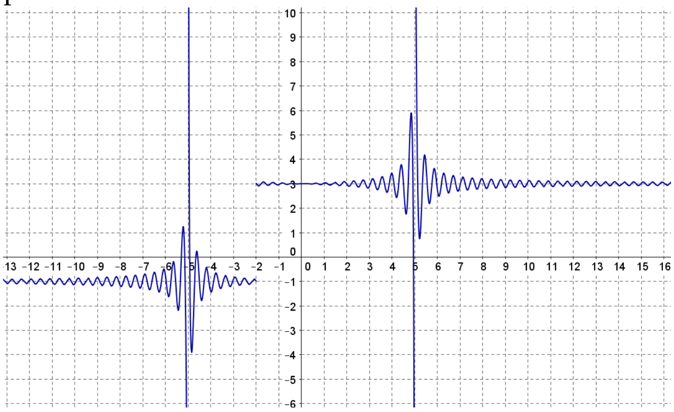
✐✠ **PRODUCTS AND FACTORS** ✠✐

# 1 Preliminary Work

## 1.1 Intuitive idea of limits

☐ **Deriv. DrH Exercise 1**
Read the limits from the graph of the function $f$ represented below.



1. $\lim_{x\to+\infty} f(x) = \ldots$
2. $\lim_{x\to-\infty} f(x) = \ldots$
3. $\lim_{x\to 0} f(x) = \ldots$
4. $\lim_{x\to-2^+} f(x) = \ldots$
5. $\lim_{x\to-2^-} f(x) = \ldots$
6. $\lim_{x\to-2} f(x) = \ldots$
7. $\lim_{x\to 5} f(x) = \ldots$

**Rule of thumb for Limits**
Limits are what they should be, meaning you can tell right away what they are, unless one of the following *inderterminate* forms appears : "$\frac{0}{0}$", "$\frac{\infty}{\infty}$", "$\infty-\infty$", "$0^0$" and "$1^\infty$".

where for example "$\frac{0}{0}$" means that the limit of the numerator is 0 and the limit of the denominator is 0 (Do not let the notation mislead you : they are NOT *equal* to 0, they *approach* 0).
Interestingly enough, when there is an indeterminate form, anything could happen : maybe the limit does not exist, maybe it does but in that case it could be any number.

**Surviving indeterminate forms**
In order to deal with the indeterminate forms, you will usually need to factorise then simplify the expression ... until it no longer is an indeterminate form.

☐ **Deriv. DrH Exercise 2**

1. $\lim_{x\to+\infty} x^2 = \ldots$
2. $\lim_{x\to-\infty} x^3 - 1 = \ldots$
3. $\lim_{x\to 0} \frac{1+x}{2-3x} = \ldots$
4. $\lim_{x\to 0^+} \frac{1}{x} = \ldots$
5. $\lim_{x\to 0^-} \frac{1}{x} = \ldots$
6. $\lim_{x\to 0} \frac{3x-4x^5}{x} = \ldots$

☐ **Deriv. DrH Exercise 3**

1. $\lim_{x\to+\infty} x^2 + 3 = \ldots$
2. $\lim_{x\to+\infty} x^2 - 5x = \ldots$
3. $\lim_{x\to-\infty} x^2 - 5x = \ldots$

☐ **Deriv. DrH Exercise 4** Evaluate $\lim_{x\to\frac{7}{2}} \frac{4x^2-49}{2x-7}$.

☐ **Deriv. DrH Exercise 5** Evaluate $\lim_{x\to-3} \frac{x^3+27}{9-x^2}$

☐ **Deriv. DrH Exercise 6**

1. $\lim_{x\to 1} \frac{x^2-1}{x+1} = \ldots$
2. $\lim_{x\to -1} \frac{x^2-1}{x+1} = \ldots$
3. $\lim_{x\to 3} \frac{x^3-8}{3x-6} = \ldots$
4. $\lim_{x\to 2} \frac{x^3-8}{3x-6} = \ldots$
5. $\lim_{x\to 2.5} \frac{2x^2-x-10}{2x-5} = \ldots$

☐ **Deriv. DrH Exercise 7**

1. $\lim_{x\to 3} \frac{x^2-9}{x+3} = \ldots$
2. $\lim_{x\to -3} \frac{x^2-9}{x+3} = \ldots$
3. $\lim_{x\to 3} \frac{x^3-27}{3x-9} = \ldots$
4. $\lim_{x\to 2} \frac{x^3-27}{3x-9} = \ldots$
5. $\lim_{x\to 2} \frac{x^3-8}{3x-6} = \ldots$
6. $\lim_{x\to 2} \frac{2x^2-x-6}{-2x+4} = \ldots$

☐ **Deriv. DrH Exercise 8** Let $f(x) = -2 - \frac{1}{x+5}$

1. Find the derivative of $f$
2. Sketch the graph of $f$

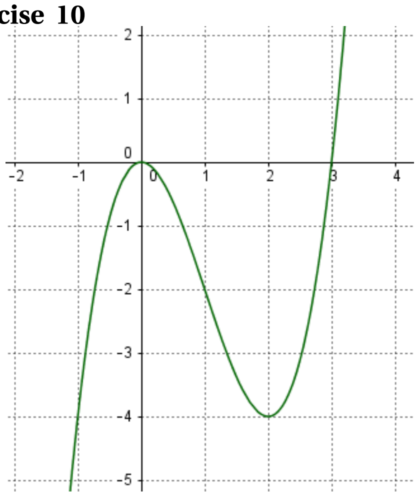
3. Find $\lim_{x\to+\infty} f(x)$
4. Find $\lim_{x\to-\infty} f(x)$
5. Find $\lim_{x\to-5^+} f(x)$
6. Find $\lim_{x\to-5^-} f(x)$

☐ **Deriv. DrH Exercise 9** Let $f(x) = 7 + \frac{1}{x-3}$

1. Find the derivative of $f$
2. Sketch the graph of $f$
3. Find $\lim_{x\to+\infty} f(x)$
4. Find $\lim_{x\to-\infty} f(x)$
5. Find $\lim_{x\to3^+} f(x)$
6. Find $\lim_{x\to3^-} f(x)$

☐ **Deriv. DrH Exercise 10**

Read the required values from the graph of the function $f$ represented on the right.




1. $f(1) \approx ...$
2. $f'(1) \approx ...$
3. $f(2) \approx ...$
4. $f'(2) \approx ...$

### 1.2 What if there is something other than $x$ in the function ?

☐ **Deriv. DrH Exercise 11** *What if there is something other than x in the function ?*
Let $f(x) = \frac{x+1}{x-1}$.
Sally says that $f\left(\frac{1}{x}\right) = -f(x)$ and Tara says that this is completely wrong. Who is right ? Give reasons (I mean a mathematical reason, not « Tara is usually right » :-).

## 2 Differenciating from First Principles

**DEGREE ONE EQUATIONS**

**"Differenciate using First Principles" ...**
... means using the *definition* of the derivative, i.e., the limit of the rate of change :

If the limit $\lim_{h\to0} \frac{f(x+h)-f(x)}{h}$ exists, then the function $f$ is *differentiable at* $x$ and the derivative at $x$ is equal to the above limit, i.e.

$$f'(x) = \lim_{h\to0} \frac{f(x+h)-f(x)}{h}$$

## 3 Differentiating powers

☐ **Deriv. DrH Exercise 12**
Find the derivative of the following functions.

1. $f_1(x) = x^7\sqrt{x}$.
2. $f_2(x) = \frac{x^7\sqrt{x}}{3}$.
3. $f_3(x) = \frac{x^4}{\sqrt{x}}$.

☐ **Deriv. DrH Exercise 13**
Let $f$ be the function defined for all $x$ by $f(x) = x^4$

1. Find the derivative of $f$.
2. Prove that the function 'looks like the square function' meaning that
   a. it is decreasing for $x < 0$
   b. it is increasing for $x > 0$
   c. and it has a horizontal tangent for $x = 0$.

☐ **Deriv. DrH Exercise 14**
Let $f$ be the function 'cube root' defined for all $x > 0$ by $f(x) = \sqrt[3]{x}$

1. Find the derivative of $f$ and rewrite it without indices.
2. Prove that the function 'cube root' is increasing on its domain.

## 4 Chain Rule

☐ **Deriv. DrH Exercise 15** Let $f$ be the function defined for all $x > 0$ by $f(x) = (\sqrt{x})^3$. The goal of this exercise is to find the derivative of $f$ using different methods.

1. Find the derivative of $f$ using the formula for differentiating a power of $x$. First rewrite $f(x) = x^\alpha$ with $\alpha = ....$
2. Find the derivative of $f$ using the Chain rule. First rewrite $f(x) = u^3$ with $u = ....$
3. Find the derivative of $f$ using the Chain rule. First rewrite $f(x) = \sqrt{u}$ with $u = ....$

☐ **Deriv. DrH Exercise 16** Let $u$ be the function defined by $u(x) = -2x^2 + 8x + 42$. Let $f$ be the function defined by $f(x) = \sqrt{u(x)}$.

1. Natural domain of $f$ :
   a. Write $u(x)$ in a factorised form.
   b. What is the natural domain of $f$ ?
2. Find the $x$-coordinate of any point where the tangent to the graph of $f$ is horizontal.

☐ **Deriv. DrH Exercise 17** Find the derivative of $f(x) = \frac{6}{(2x+5)^3}$

## 5 Product rule

☐ **Deriv. DrH Exercise 18** Let $f$ be the function defined by $f(x) = -3x\sqrt{1-4x}$.

1. Find the natural domain of $f$.
2. Differentiate $f$.
3. Find the point(s) where the tangent to the graph of $f$ is horizontal.

## 6 Quotient rule

## 7 Differentiability

☐ **Deriv. DrH Exercise 19** Let $f$ be the function defined by $f(x) = \begin{cases} (x-2)^2 & \text{if } x \leq 3 \\ 2-(x-4)^2 & \text{if } x > 3 \end{cases}$

1. Sketch the graph of $f$.
2. Is $f$ continuous at $x = 3$?
3. Is $f$ differentiable at $x = 3$?

## 8 Tangent, Normal

☐ **Deriv. DrH Exercise 20** Is it possible to chose $p$ such that the tangent to $y = 3x^2 + px + 7$ at the point with $x$-coordinate equal to 2 is parallel to the straight line $\ell$ with equation $5x - y + 8 = 0$?

*In case you were wondering, this was written using* LaTeX *(free software for beautiful documents).*

***Solution of Deriv. DrH Exercise 1 .***

***Solution of Deriv. DrH Exercise 2 .***

***Solution of Deriv. DrH Exercise 3 .***

***Solution of Deriv. DrH Exercise 4 .***

$\frac{4x^2-49}{2x-7} = 2x+7$ so $\lim_{x\to\frac{7}{2}} \frac{4x^2-49}{2x-7} = 2\frac{7}{2}+7 = 14$

***Solution of Deriv. DrH Exercise 5 .***

$\frac{x^3+27}{9-x^2} = -\frac{x^2-3x+9}{x-3}$ so $\lim_{x\to-3} \frac{x^3+27}{9-x^2} = -\frac{(-3)^2-3(-3)+9}{-3-3} = -\frac{9}{2}$

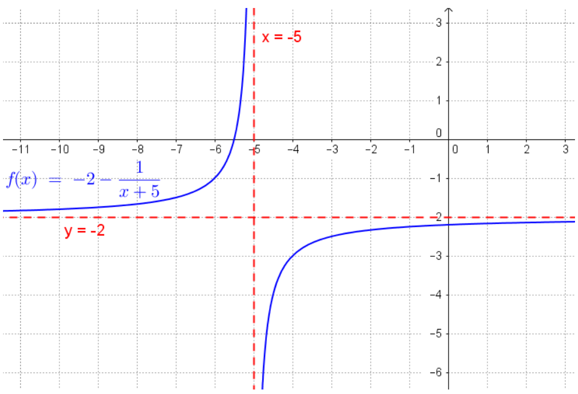
***Solution of Deriv. DrH Exercise 6 .***

***Solution of Deriv. DrH Exercise 7 .***

1. $\lim_{x\to3} \frac{x^2-9}{x+3} = \ldots$
2. $\lim_{x\to-3} \frac{x^2-9}{x+3} = \ldots$
3. $\lim_{x\to3} \frac{x^3-27}{3x-9} = \ldots$
4. $\lim_{x\to2} \frac{x^3-27}{3x-9} = \ldots$
5. $\frac{x^3-8}{3x-6} = \frac{x^2+2x+4}{3}$ so $\lim_{x\to2} \frac{x^3-8}{3x-6} = 8$
6. Simplify : $\frac{2x^2-x-6}{-2x+4} = -\frac{2x+3}{2}$ so $\lim_{x\to2} \frac{2x^2-x-6}{-2x+4} = -\frac{7}{2}$

***Solution of Deriv. DrH Exercise 8 .***

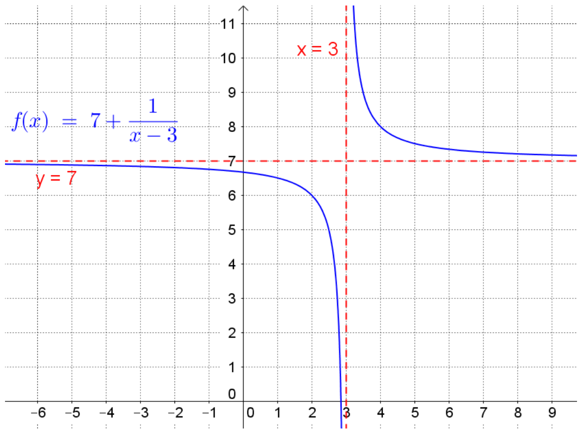
1. $f(x) = -2 - \frac{1}{x+5} = -2-(x+5)^{-1}$ so its derivative is $f'(x) = 0 - 1\times -1\times(x+5)^{-2} = \frac{1}{(x+5)^2}$
2. Graph of $f$




3. $\lim_{x\to+\infty} f(x) = -2$
4. $\lim_{x\to-\infty} f(x) = -2$
5. As $x$ approaches $-5$ from the right, $f(x)$ becomes infinitely large in absolute value (and negative) so the limit is $-\infty$. $\lim_{x\to-5^+} f(x) = -\infty$
6. As $x$ approaches $-5$ from the left, $f(x)$ becomes infinitely large (and positive) so the limit is $+\infty$. $\lim_{x\to-5^-} f(x) = +\infty$.

***Solution of Deriv. DrH Exercise 9 .***

1. $f(x) = 7+\frac{1}{x-3} = 7+(x-3)^{-1}$ so its derivative is $f'(x) = 0-1\times(x-3)^{-2}\times1 = -\frac{1}{(x-3)^2}$
2. Graph of $f$




3. $\lim_{x\to+\infty} f(x) = 7$
4. $\lim_{x\to-\infty} f(x) = 7$
5. As $x$ approaches 3 from the right, $f(x)$ becomes infinitely large in absolute value (and positive) so the limit is $+\infty$. $\lim_{x\to-5^+} f(x) = +\infty$
6. As $x$ approaches 3 from the left, $f(x)$ becomes infinitely large (and negative) so the limit is $-\infty$. $\lim_{x\to3^-} f(x) = -\infty$.

***Solution of Deriv. DrH Exercise 10 .***

***Solution of Deriv. DrH Exercise 11 .***

$f\left(\frac{1}{x}\right) = \frac{\frac{1}{x}+1}{\frac{1}{x}-1} = \frac{1+x}{1-x}$ (multiply the numerator and the denominator by $x$).
Now note that $\frac{1+x}{1-x} = \frac{x+1}{-(x-1)} = -\frac{x+1}{x-1} = -f(x)$ so Sally is right.

***Solution of Deriv. DrH Exercise 12 .***

1. $f_1(x) = x^7\sqrt{x} = x^7x^{\frac{1}{2}} = x^{7+\frac{1}{2}} = x^{\frac{15}{2}}$.
   Therefore, $f_1'(x) = \frac{15}{2}x^{\frac{15}{2}-1} = \frac{15}{2}x^{\frac{13}{2}} = \frac{15}{2}x^{\frac{12}{2}+\frac{1}{2}} = \frac{15}{2}x^{\frac{12}{2}}x^{\frac{1}{2}} = \frac{15}{2}x^6\sqrt{x}$.
2. $f_2(x) = \frac{x^7\sqrt{x}}{3} = \frac{1}{3}x^7\sqrt{x} = \frac{1}{3}f_1(x)$.
   Therefore, $f_2'(x) = \frac{1}{3}f_1'(x) = \frac{1}{3}\times\frac{15}{2}x^6\sqrt{x} = \frac{5}{2}x^6\sqrt{x}$
3. $f_3(x) = \frac{x^4}{\sqrt{x}} = \frac{x^4}{x^{\frac{1}{2}}} = x^{4-\frac{1}{2}} = x^{\frac{7}{2}}$.
   Therefore, $f_3'(x) = \frac{7}{2}x^{\frac{7}{2}-1} = \frac{7}{2}x^{\frac{5}{2}} = \frac{7}{2}x^{\frac{4}{2}+\frac{1}{2}} = \frac{7}{2}x^2x^{\frac{1}{2}} = \frac{7}{2}x^2\sqrt{x}$

***Solution of Deriv. DrH Exercise 13 .***

Un jour....

***Solution of Deriv. DrH Exercise 14 .***

Un jour....

***Solution of Deriv. DrH Exercise 15 .***

1. $f(x) = (\sqrt{x})^3 = x^{\frac{3}{2}}$, so $f(x) = x^\alpha$ with $\alpha = \frac{3}{2}$ .Therefore its derivative is $f'(x) = \frac{3}{2}x^{\frac{3}{2}-1} = \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}$

2. Find the derivative of $f$ using the Chain rule. First rewrite $f(x) = u^3$ with $u = \sqrt{x}$. Therefore $f'(x) = 3u^2 \times \frac{1}{2\sqrt{x}} = 3\sqrt{x}^2 \times \frac{1}{2\sqrt{x}} = 3x \times \frac{1}{2\sqrt{x}} = \frac{3x}{2\sqrt{x}} = \frac{3x\times\sqrt{x}}{2\sqrt{x}\times\sqrt{x}} = \frac{3}{2}\sqrt{x}$

3. $f(x) = (\sqrt{x})^3 = x^{\frac{3}{2}} = \left(x^3\right)^{\frac{1}{2}} = \sqrt{x^3} = \sqrt{u}$ with $u = x^3$. Therefore $f'(x) = \frac{1}{2\sqrt{u}} \times 3x^2 = \frac{1}{2\sqrt{x^3}} \times 3x^2 = \frac{3x^2}{2\sqrt{x^3}} = \frac{3x^2}{2x^{\frac{3}{2}}} = \frac{3x^2x^{-\frac{3}{2}}}{2} = \frac{3x^{2-\frac{3}{2}}}{2} = \frac{3x^{\frac{1}{2}}}{2} = \frac{3\sqrt{x}}{2} = \frac{3}{2}\sqrt{x}$.

Of course, we get the same derivative each and every time.

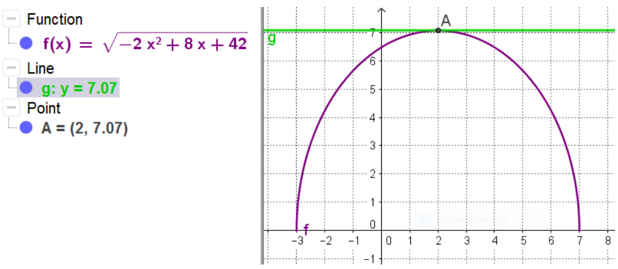
***Solution of Deriv. DrH Exercise 16 .***

1. Natural domain of $f$ :

 a. $u(x) = -2(x+3)(x-7)$

 b. The natural domain of $f$ is all the values of $x$ such that $u(x) \geqslant 0$. i.e. $-3 \leqslant x \leqslant 7$. (Graph $u$, using the $x$-intercepts found in part 1).

 $f(x) = \sqrt{u(x)}$ with $u(x) = -2x^2+8x+42$ so its derivative is $f'(x) = \frac{1}{2\sqrt{u}} \times (-4x+8) = \frac{-2x+4}{\sqrt{-2x^2+8x+42}}$. The $x$-coordinate of any point where the tangent to the graph of $f$ is horizontal is a solution of $f'(x) = 0$ i.e. $x = 2$.



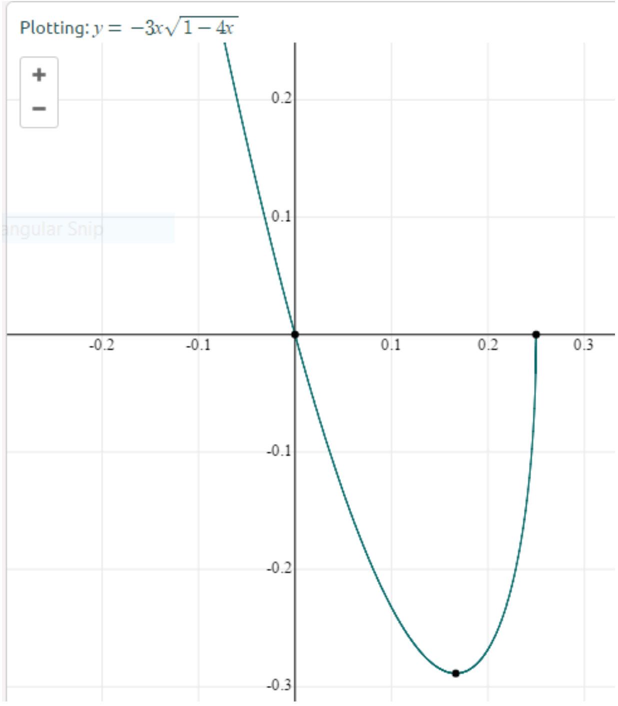

2. ***Solution of Deriv. DrH Exercise 17 .***
$\frac{d}{dx}\left(\frac{6}{(2x+5)^3}\right) = 6\frac{d}{dx}\left(\frac{1}{(2x+5)^3}\right) = 6\frac{d}{dx}\left((2x+5)^{-3}\right) = 6\frac{d}{du}\left(u^{-3}\right)\frac{d}{dx}(2x+5)$ with $u = 2x+5$.
$f'(x) = 6 \times -3u^{-4} \times 2 = -36(2x+5)^{-4} = -\frac{36}{(2x+5)^4}$.

***Solution of Deriv. DrH Exercise 18 .***

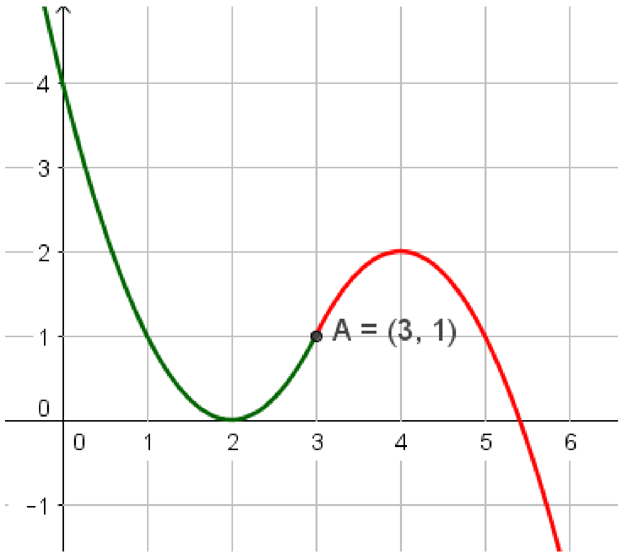
1. Natural domain of $f$ :

2. $f'(x) = \frac{d}{dx}\left(-3x\sqrt{1-4x}\right) = \frac{18x-3}{\sqrt{-4x+1}}$

3. The $x$-coordinate of any point where the tangent to the graph of $f$ is horizontal is a solution of $f'(x) = 0$. $f'(x) = 0 \iff 18x - 3 = 0 \iff x = \frac{1}{6}$. The $y$-coordinate of the point is $y = f\left(\frac{1}{6}\right) = -3 \times \frac{1}{6}\sqrt{1 - 4 \times \frac{1}{6}} = -\frac{1}{2}\sqrt{\frac{1}{3}} = -\frac{\sqrt{3}}{6}$
 ANSWER : The tangent to the the graph of $f$ is horizontal at the point with coordinates $\left(\frac{1}{6}, -\frac{\sqrt{3}}{6}\right)$




***Solution of Deriv. DrH Exercise 19 .***

1. 


2. $\lim_{x\to3^-} f(x) = 1$ and $\lim_{x\to3^+} f(x) = 1$ . They are equal so $f$ is continuous at $x = 3$. This is what we expected from the diagram as there is no break in the graph.

3. Strategy : Find the derivative on both sides of 3 and take their limit.
 $\lim_{x\to3^-} f'(x) = 2$ and $\lim_{x\to3^+} f'(x) = 2$. They are equal so $f$ is differentiable at $x = 3$ with $f'(3) = 2$. This is what we expected from the diagram as there is no break and no sharp corner in the graph.

***Solution of Deriv. DrH Exercise 20 .***
$5x - y + 8 = 0 \iff y = 5x + 8$ so the gradient of $\ell$ is 5.
$\frac{dy}{dx} = 6x + p$ so $f'(2) = 12 + p$.
Parallel lines have the same gradient so $12 + p = 5$ so $p = -7$.