Summary	TANGENT and the DERIVATIVE	11MAD

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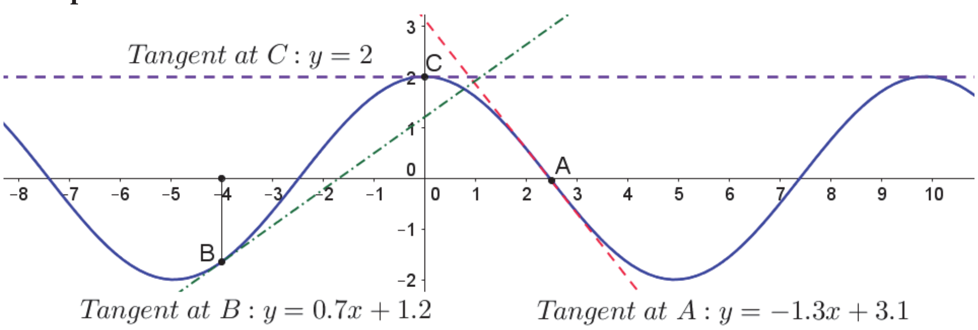
Derivatives measure change : If a quantity increases, its derivative will be positive. If a quantity decreases, its derivative will be negative. Even better, if two quantities are increasing, the one which increases faster will have a larger derivative. In short, the derivative tells you if a quantity is increasing or decreasing, and how fast this is happening.

➫ It turns out that the gradient of the tangent to a curve does exactly that, so we will define the derivative of a function to be the the gradient of the tangent to its graph.

I The derivative as the gradient of the tangent to a curve

Geometric Interpretation of the derivative : It is the gradient of the tangent.
The *derivative* of a function f at x, denoted $f'(x)$, is the gradient of the tangent at the point of the curve with x-coordinate equal to x.

Example 1.



By definition, $f'(-4)$, is the gradient of the tangent at the point of the curve with x-coordinate equal to -4, so $f'(-4) = 0.7$. Similarily, $f'(2.5) = -1.3$ and $f'(0) = 0$.

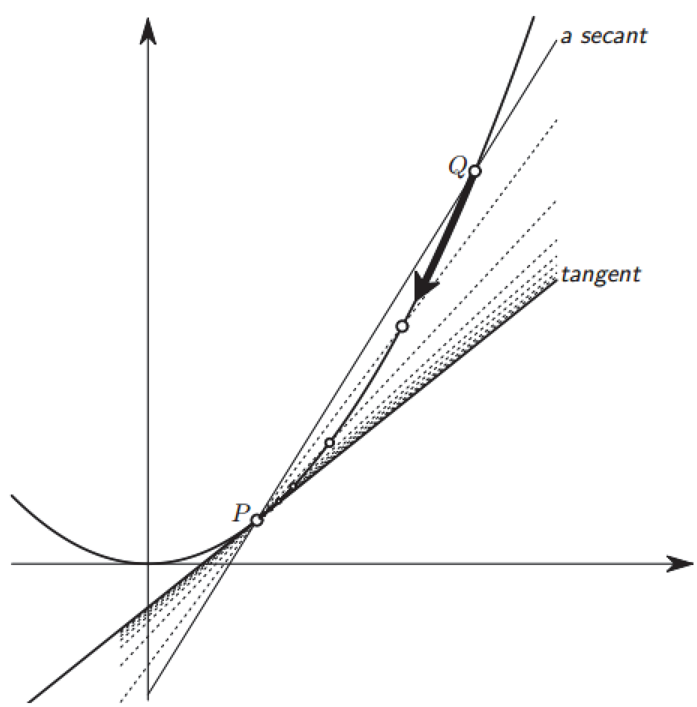
Remark 1. In particular, this gives the derivative of any constant or linear function, because they are their own tangents :

Constant functions	If $f(x) = k$ for some number k for all x	then $f'(x) = 0$ for all x
Linear functions	If $f(x) = mx + b$ for some numbers m and b for all x	then $f'(x) = m$ for all x

Nice, but...how do we find the gradient of the tangent ?

Suppose you have a function $y = f(x)$ and you draw its graph. If you want to find the tangent to the graph of f at some given point on the graph of f, how would you do that ?

Let P be the point on the graph at which want to draw the tangent. Pick a point Q on the graph and construct the line through P and Q. This line is called a "secant," and it is of course not the tangent that you're looking for. But if you choose Q to be very close to P then the secant will be close to the tangent.
So this is our recipe for constructing the tangent through P : pick another point Q on the graph, find the line through P and Q, and see what happens to this line as you take Q closer and closer to P. The resulting secants will then get closer and closer to some line, and that line is the tangent.

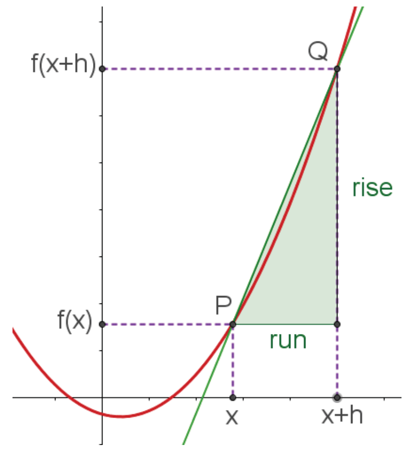


✎ Let's do it! and remember that our goal is to find $f'(x)$, the gradient of the tangent at P.

If the coordinates of P are $(x, f(x))$ and the coordinates of Q are $(x+h, f(x+h))$, then
The gradient of the secant $PQ = \frac{rise}{run} = \frac{y_Q - y_P}{x_Q - x_P} = \frac{f(x+h)-f(x)}{x+h-x} = \frac{f(x+h)-f(x)}{h}$
As Q approaches P (which is the same as saying that h approaches 0) the gradient of the secant PQ approaches the gradient of the tangent, denoted $f'(x)$.
With limit notations, we rewrite the above sentence as :

$$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = f'(x)$$

Read : *"The limit as h approaches 0 of $\frac{f(x+h)-f(x)}{h}$ is equal to $f'(x)$."*
➭ We adopt this as the *definition* of the derivative.



(Average) rate of Change : The quotient $\frac{f(x+h)-f(x)}{h} = \frac{Change\ in\ y}{Change\ in\ x}$ is called the *(average) rate of change*[1] of f with respect to x over the interval x to $x+h$. It is sometimes also denoted $\frac{\Delta y}{\Delta x}$ or $\frac{\delta y}{\delta x}$ where Δx (or δx) is the change in x and Δy (or δy) is the corresponding change in y.

E.g. : The average velocity, given by the formula $v = \frac{distance\ travelled}{time\ it\ took}$ is an average rate of change. The *instantaneous* velocity is the one you read on the speedometer of your car. It is obtained by evaluating the average velocity over smaller and smaller intervals of times... so the instantaneous velocity is a derivative! (the derivative of position)

II The derivative as the Limit of a Rate of Change

Definition of the derivative : It is the limit of the rate of change.
If the limit $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists (meaning that $\frac{f(x+h)-f(x)}{h}$ approaches a finite number as h approaches 0), then the function f is *differentiable at* x and the derivative at x, denoted $f'(x)$, is equal to the above limit, i.e.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} \tag{1}$$

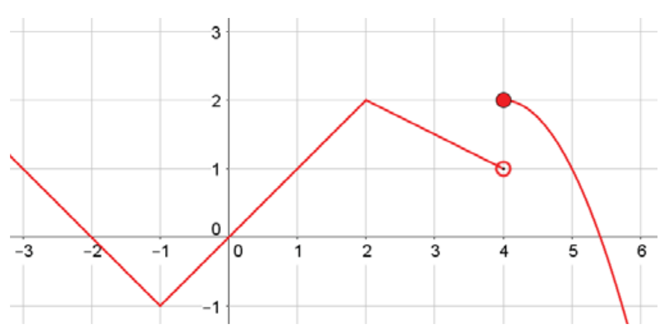
Remark 2. *"Differenciate using First Principles"* means using the *definition* of the derivative, i.e., the limit of the rate of change as given in the definition above.

Practice : In order to differentiate using first principles

1. Write and simplify the rate of change $\frac{f(x+h)-f(x)}{h}$,
2. then determine if the limit of the rate of change exists as h approaches 0.

Exercise 1. Differentiate from first principles $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{x}$. Challenge : What about $h(x) = x^3$?

Remark 3. If the graph has a sharp corner or if there is a *discontinuity* (=break) in the graph, then the limit of the rate of change doesn't exist at that point so the function is not differentiable for such value of x.
It makes sense because there is no tangent at those points so no gradient of the tangent either!



This function is differentiable neither at $x = -1$, nor at $x = 2$ (sharp corner), nor at $x = 4$ (discontinuity *in the graph).*

1. The Greek letter *delta* (written Δ in upper case and δ in lower case) is often used in mathematics and physics to mean 'change'.

⚠ This will be something to look out for if the function is defined piecewise (as the one above) :

If the function is defined piecewise ('piecemeal' on this side of the equator), you will need to look at the limits of the rate of change when h approaches 0 from the left and from the right separately. If the both limits exist and if they are equal, then the function is differentiable at x and its derivative $f'(x)$ is the common value of the two limits.

An alternative notation for the derivative : Liebniz's notation.

If we denote the derivative of y with respect to x by $\frac{dy}{dx}$, the definition $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ can be rewritten

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} \tag{2}$$

where $\delta y = f(x+\delta x) - f(x)$ is the small change in y resulting from a small change δx in x.

Remark 4. The derivative $\frac{dy}{dx}$ is NOT a fraction, Liebniz's notation just reminds us that is is the *limit* of the fraction $\frac{\delta y}{\delta x}$ (and it helps us remember the chain rule, see below).

Remark 5. $\frac{f(x+\delta x)-f(x)}{\delta x}$ is the *average* rate of change of f over the interval from x to $x+\delta x$. Its limit as δx approaches zero, the derivative $\frac{dy}{dx}$, is the *instantaneous* rate of change of f.

III Rules for Differentiating

Derivative and Operations (Sum, product..etc)

DERIVATIVE OF A SUM	If $f(x) = g(x) + h(x)$	then $f'(x) = g'(x) + h'(x)$ also written $f' = g' + h'$
DERIVATIVE OF A MULTIPLE	If $f(x) = a \times g(x)$ for some number a	then $f'(x) = a \times g'(x)$ also written $f' = ag'$
DERIVATIVE OF A PRODUCT *PRODUCT RULE*	If $f(x) = u(x) \times v(x)$	then $f'(x) = v(x) \times u'(x) + u(x) \times v'(x)$ also written $f' = vu' + uv'$ or $\frac{dy}{dx} = \frac{u}{dx} + u\frac{dv}{dx}$
DERIVATIVE OF A QUOTIENT *QUOTIENT RULE*	If $f(x) = \frac{u(x)}{v(x)}$	then $f'(x) = \frac{v(x)\times u'(x) - u(x)\times v'(x)}{(v(x))^2}$ also written $f' = \frac{vu' - uv'}{v^2}$ or $\frac{dy}{dx} = \frac{v\frac{u}{dx} - u\frac{dv}{dx}}{v^2}$

Differentiating powers : Take the index as a factor and reduce the index by 1.

The derivative of the function defined by $f(x) = x^n$ for any real number n is $f'(x) = nx^{n-1}$

This wonderful formula includes all the cases you need to know so far :

$n = 0$	If $f(x) = 1$ for all x	then $f'(x) = 0$ for all x
$n = 1$	If $f(x) = x$ for all x	then $f'(x) = 1$ for all x
$n = \frac{1}{2}$	If $f(x) = \sqrt{x}$ for all $x \geqslant 0$	then $f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$ for all $x > 0$
$n = -1$	If $f(x) = x^{-1} = \frac{1}{x}$ for all $x \neq 0$	then $f'(x) = -x^{-2} = -\frac{1}{x^2}$ for all $x \neq 0$

Combined with the rules about differentiating a sum an a constant multiple, and you can differentiate any polynomial. In particular, this gives another way of finding the derivative of any constant or linear function :

Constant functions	If $f(x) = k$ for some number k for all x	then $f'(x) = 0$ for all x
Linear functions	If $f(x) = mx + b$ for some numbers m and b for all x	then $f'(x) = m$ for all x

Differentiating when there is a function inside another function : The Chain Rule

✎ If $f(x) = g(u(x))$ then $f'(x) = g'(u(x)) \times u'(x)$

'When there is a function inside another function, the derivative is the derivative of the outside function evaluated at the inside function, times the derivative of the inside function'.

✎ **Using Liebniz's notation** : If y is a function of u and u is a function of x, then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.

Example 2. : Let $f(x) = (2x+3)^7$: $f(x)$ is of the form $f(x) = g(u(x))$ with $g(x) = x^7$ and $u(x) = 2x+3$. g is the 'outside function' and u is the 'inside function'. This can be represented by : $x \overset{u}{\mapsto} 2x+3 \overset{g}{\mapsto} f(x) = (2x+3)^7$. Let's find its derivative : $g'(x) = 7x^6$ and $u'(x) = 2$ so, with the Chain Rule, we get $f'(x) = g'(u(x)) \times u'(x) = 7(2x+3)^6 \times 2$.

Remark 6. Don't expand! We are usually interested in the sign of the derivative because a positive derivative tells us that the function is increasing and a negative derivative tells us that the function is decreasing. Since it is much easier to figure out the sign when the expression is factorised, we don't usually expand derivatives.

IV Tangent and Normal to a curve

Let a be a number in the domain of a differentiable function f, and let $A(a, f(a))$ be the point with x-coordinate equal to a lying on the curve of f.

Use the *point-gradient* form [2] of the equation of a line to find the equation of the tangent and the normal to a curve at a given point :

Tangent and Normal to a curve.

1. The tangent at A is the line through A whose gradient is the derivative at a, i.e. $m_T = f'(a)$.
   The equation of the tangent at A is therefore :
   $y = f'(a)(x-a) + f(a)$.
2. The *normal* at A is the line through A which is perpendicular to the tangent at a. Its gradient [a] is therefore $m_\perp = -\frac{1}{f'(a)}$.
   The equation of the normal at A is therefore :
   $y = -\frac{1}{f'(a)}(x-a) + f(a)$.

a. The symbol $\perp$ means *perpendicular*.

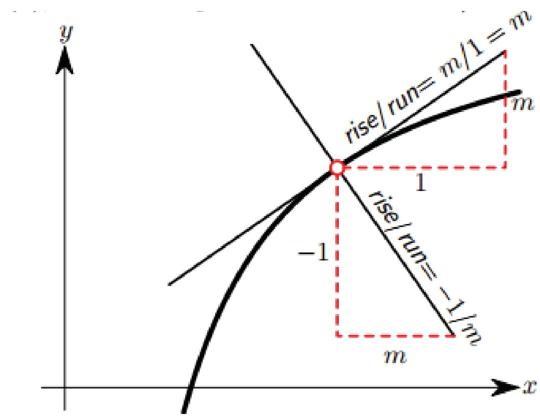


Figure 1. Why "slope of normal $= \dfrac{-1}{\text{slope of tangent}}$."

V Basics Notions about Limits and Continuity

Continuity

Intuitively, a function is *continuous* if its graph can be drawn without lifting the pen. In other words, there are no discontinuity (=no break) in the graph. (see figure at bottom of page 2).

Rule of thumb for Limits

Limits are what they should be, meaning you can tell right away what they are, unless one of the following *inderterminate* forms appears : "$\frac{0}{0}$", "$\frac{\infty}{\infty}$", "$\infty - \infty$", "0^0" and "1^∞".

where for example "$\frac{0}{0}$" means that the limit of the numerator is 0 and the limit of the denominator is 0 (Do not let the notation mislead you : they are NOT *equal* to 0, they *approach* 0).

Interestingly enough, when there is an indeterminate form, anything could happen : maybe the limit does not exist, maybe it does but in that case it could be any number.

2. The equation of the line through (x_1, y_1) with gradient m is : $y - y_1 = m(x - x_1)$, which can be rewritten $y = m(x - x_1) + y_1$.

Surviving indeterminate forms

In order to deal with the indeterminate forms, you will usually need to factorise then simplify the expression ... until it no longer is an indeterminate form.

How to tell if a function is continuous at a point?

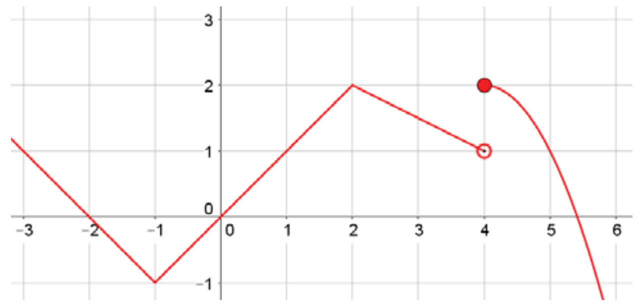
A function f is continuous at $x = a$ if $\lim_{x \to a^+} f(x)$, $\lim_{x \to a^-} f(x)$ and $f(a)$ all exist and are equal.

How do I use this? Find the limit of $f(x)$ when x approaches a from the right, find the limit of $f(x)$ when x approaches a from the left, find $f(a)$ and compare these three numbers (if they exist). If they are equal, your function is continuous at $x = a$.

Example 3. In the example on the right, with $a = 4$:

- $\lim_{x \to 4^+} f(x) = 2$
- $\lim_{x \to 4^-} f(x) = 1$
- $f(a) = 2$

  These numbers are NOT all equal so the function is not continuous at $x = 4$ (which we knew because we can see the "break" in the graph.) We can also say that there is a *discontinuity* in the graph at $x = 4$

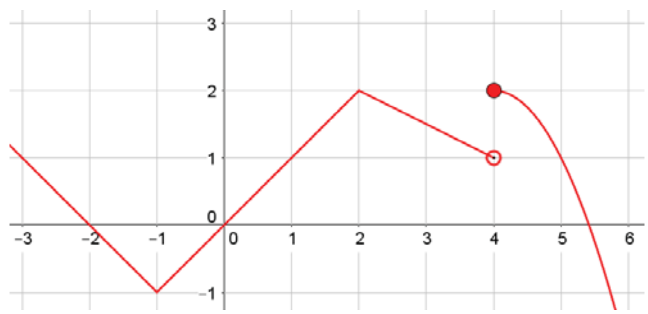


VI Differentiability

As already mentioned in Remark 3 , if the graph has a sharp corner or if there is a *discontinuity* (=break) in the graph, then the limit of the rate of change doesn't exist at that point so the function is not differentiable for such value of x.
It makes sense because there is no tangent at those points so no gradient of the tangent either!

It is in not continuous, it can NOT be differentiable!

If a function is NOT continuous at $x = a$, it is certainly not differentiable there.



This function is differentiable neither at $x = -1$, nor at $x = 2$ (sharp corner), nor at $x = 4$ (discontinuity *in the graph).*

How to tell if a (piecewise-defined) function is differentiable at a point?

If a function f is continuous at $x = a$ and $\lim_{x \to a^+} f'(x)$ and $\lim_{x \to a^-} f'(x)$exist and are equal

then the function is differentiable at $x = a$ and $f'(x) = \lim_{x \to a^+} f'(x) = \lim_{x \to a^-} f'(x)$

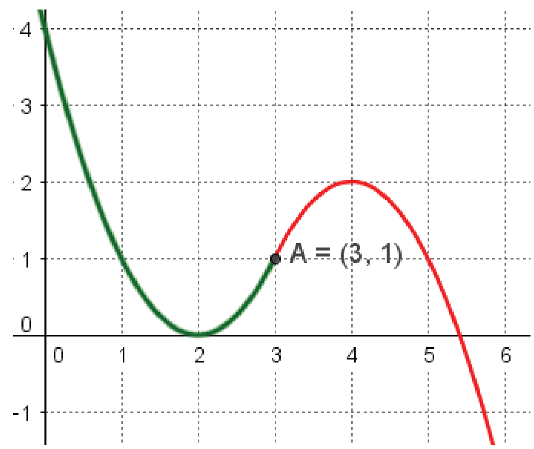
How do I use this?

- First, check if the function is continuous at $x = a$. If it is not, you are done, as the function cannot be differentiable either. If it continuous at $x = a$, keep going :
- Find the derivative of the function for $x > a$ and then take its limit as x approaches a from the right.
- Find the derivative of the function for $x < a$ and then take its limit as x approaches a from the left.
- If these two numbers are the same, your function is differentiable at $x = a$.

Example 4. Let $f(x) = \begin{cases} (x-2)^2 & \text{if } x < 3; \\ -(x-4)^2 + 2 & \text{if } x \geqslant 3. \end{cases}$

Is f differentiable at $x = 3$?

From its graph, we guess that f is continuous at $x = 3$ as there is not break in the graph and that it is differentiable at $x = 3$ as there is no sharp corner. Let's prove it.



Step 1. Is f continuous at $x = 3$?

- $\lim_{x\to 3^-} f(x) = (3-1)^2 = 1$
- $\lim_{x\to 3^+} f(x) = -(3-4)^2 + 2 = -1 + 2 = 1$
- $f(1) = -(3-4)^2 + 2 = -1 + 2 = 1.$
  These three numbers are all equal so the function is continuous at $x = 3$ (meaning we must keep going).

Step 2. Is f differentiable at $x = 3$?

- If $x < 3$, $f'(x) = 2x - 4$ so $\lim_{x\to 3^-} f'(x) = 2 \times 3 - 4 = 2.$
- If $x > 3$, $f'(x) = -2x+8$ so $\lim_{x\to 3^+} f'(x) = -2\times 3+8 = -6+8 = 2.$
  These two numbers are all equal so f differentiable at $x = 3$ and $f'(3) = 2.$