UNSW Mathematics Teachers PD Day

Vectors, lines and projectile motion

(ME-V1, MEX-V1)

Dr H (Laure Helme-Guizon)

l.helme-guizon@unsw.edu.au

School of Mathematics and Statistics
University of New South Wales, Sydney

Please, install the KAHOOT app on your **phone**!
(Or play on your laptop.)

Vectors in the new syllabus [while you get ready for KAHOOT]

ME-V1 Introduction to Vectors [Extension 1, new syllabus, Y12]

1. Introduction to two-dimensional vectors: direction, magnitude.
2. Addition and subtraction of vectors (triangle law and parallelogram law), multiplication by a scalar.
3. Scalar (dot) product, expressed in terms of coordinates or $cos\theta$, expression for the magnitude of a vector, parallel and perpendicular vectors, projection of one vector onto another.
4. Proofs in 2D using vectors, unit vectors.
5. Solve problems involving displacement, force and velocity involving vector concepts in two dimensions + Projectile motion.

MEX-V1 Further Work with Vectors [Extension 2, new syllabus, Y12]

1. Extend the above concepts to three-dimensional vectors. ~~Cross Product~~. Proofs in 2D and 3D using vectors.
2. Vectors and vector equations of lines in 2D and 3D, parallel and perpendicular lines in 2D and 3D.
3. Complex numbers may be represented using polar coordinates or as vectors. [...] Addition and subtraction of complex numbers as vectors in the complex plane

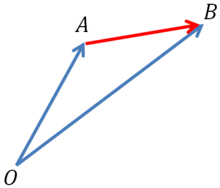
Warm up on vectors via a KAHOOT game

Here it is: Kahoot PD Day, Warm up on Vectors

Download the KAHOOT app on you phone or turn on your laptop so you can play the game!

Position vs Displacement Vector

Consider an origin O shown below.
Consider a particle that moves from point A to point B.



• The position vector of a particle is defined as the vector starting from the origin to the point where the particle is.

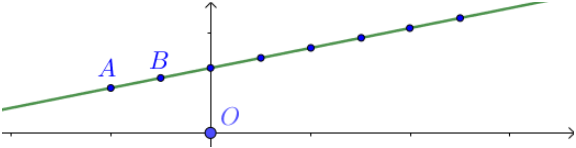
In the above diagram, the position vector of the particle when it is at point A is the vector $\overrightarrow{OA}$ and when it is at the point B is $\overrightarrow{OB}$.

• The displacement vector of a particle is defined as the vector joining its initial position to its final position.

In other words, the displacement vector is a *change in position* vector.

Describing a line using a point and a vector

Example 1: A car is going up a hill at a constant speed. At $t = 0$, it is at point A and after one second, it is at point B.



1. After 3 seconds, the car is at a point P_3 whose position is given by :

$$\overrightarrow{OR_3} = \overrightarrow{OA} + \dots \overrightarrow{AB}$$

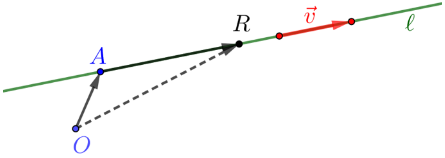
2. After 5 seconds, the car is at a point P_5 whose position is given by :

$$\overrightarrow{OR_5} = \overrightarrow{O\dots} + \dots \overrightarrow{AB}$$

3. After t seconds, the car is at a point P_t whose position is given by :

$$\overrightarrow{OR_t} = \overrightarrow{O\dots} + \dots \overrightarrow{AB}$$

Vector Equation of a Straight line



(same definition in 2D, 3D ... and many-D)
Select a point A on the line ℓ, and a non-zero vector $\vec{v}$ parallel to the line.

A point R is on the line ℓ if and only if $\overrightarrow{AR} = \lambda\vec{v}$ for some real number λ.

Since $\overrightarrow{OR} = \overrightarrow{OA} + \overrightarrow{AR}$, this can be rewritten $\overrightarrow{OR} = \overrightarrow{OA} + \lambda\vec{v}$, or $\vec{r} = \vec{a} + \lambda\vec{v}$ where $\vec{r} := \overrightarrow{OR}$ and $\vec{a} := \overrightarrow{OA}$

- The equation $\boxed{\vec{r} = \vec{a} + \lambda\vec{v} \quad (\lambda \in \mathbb{R})}$ is called a *vector equation* of the line ℓ.

- Each value of the parameter λ determines a unique point R on the line ℓ, with position vector $\vec{r} = \vec{a} + \lambda\vec{v}$. As λ takes all possible values, R takes all possible positions on the line λ.

- The equation is not unique, as a different point A on the line could have been chosen, and $\vec{v}$ can be replaced by any other non-zero vector parallel to ℓ.

Vector Equation of a line

Example 2 (Together): Give an equation, in parametric vector form, of the line ℓ through the point $A(1,-3)$ and parallel to vector $\vec{v} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$

This is like taking the train:

1. Go from home (the origin) to a train station (a point on the line)
2. and then, move along the tracks!

Vector Equation of a line

Your turn! Exercise 3: Give an equation, in parametric vector form, of the line ℓ through the point $A(-7, 3)$ parallel to vector $\vec{v} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$

Bonus: Is this equation unique? If not, find two other equations of this line.

2D: From parametric to Cartesian form

Exercise 4: Let ℓ be the line defined by $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -5 \end{pmatrix}$, for $\lambda \in \mathbb{R}$

Give an equation of ℓ in Cartesian form.

2D: From parametric to Cartesian form

Exercise 4: Let ℓ be the line defined by $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -5 \end{pmatrix}$, for $\lambda \in \mathbb{R}$

Give an equation of ℓ in Cartesian form.

ANSWER: $5x + 2y = -9$ is an equation for this line (Eliminate λ between the two equations.)

2D: From parametric to Cartesian form

Exercise 5 (Harder): Let ℓ be the line defined by

$$\vec{r} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3\sqrt{2} \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ \sqrt{2} \end{pmatrix}, \text{for } \lambda \in \mathbb{R}$$

Give an equation of ℓ in Cartesian normal form.

2D: From parametric to Cartesian form

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Give an equation of ℓ in Cartesian normal form.

ANSWER: $x - \sqrt{2}y = 2\sqrt{2}$ is an equation for this line in Cartesian normal form.
$\sqrt{2}x - 2y = 8$ is an another one.

The other way round: From Cartesian to parametric

Exercise 6: Let ℓ be the line defined by : $x + 5y = 7$
Give an equation of ℓ in parametric vector form.

The other way round: From Cartesian to parametric

Exercise 6: Let ℓ be the line defined by : $x + 5y = 7$
Give an equation of ℓ in parametric vector form.

ANSWER: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ 1 \end{pmatrix}$, for $\lambda \in \mathbb{R}$
is an equation for this line in parametric vector form.
(To get this, we used y as a parameter and then renamed it λ).

In case you haven't been impressed so far ...

Question 7: What kind of equation(s) could we use to describe a line in dimension 3?

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1. Cartesian? A line as the intersection of two planes. It works but ...

   Example: How will you be able to tell if the line ℓ defined by
   $\ell : \begin{cases} x + 3y - 2z = 4 \\ -3x + y + z = -2 \end{cases}$
   is parallel to a given line? or perpendicular to some plane? or find its intersection with some other line?

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2. Parametric? YES!!

Vectors and Derivatives

Position, Velocity and Acceleration Vectors

If a particle moves along a time-parameterised curve, its position is given by its position vector

$$\overrightarrow{OR} = \vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Its velocity vector is

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}.$$

Its acceleration vector is

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \begin{pmatrix} x''(t) \\ y''(t) \end{pmatrix}.$$

with obvious generalisation for three dimensions.

N.B. Velocity is a Vector whereas *speed*, which is the *magnitude* or *norm* of velocity, is a real number. Being a magnitude, speed cannot be negative.

Exercise 8:

The position of an object at a given time t is given by
$\overrightarrow{OR} = \vec{r}(t) = \begin{pmatrix} cos(2\pi t) \\ sin(2\pi t) \end{pmatrix}$, with $x(t)$ and $y(t)$ in metres and t in seconds.

1. What is the velocity at time t?
2. Show that the object moves at a constant speed.
3. Find the equation of the trajectory (i.e. the path of the moving object)
4. Show that the acceleration vector and the position vector are parallel.

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This is called *uniform circular motion*.

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NEWTONS LAWS OF MOTION

... really help understand what is going on!

Extension 1 new syllabus, page 51

• Solve problems involving displacement, force and velocity involving vector concepts in two dimensions

Extension 2 new syllabus, page 42

• Use Newton's laws to obtain equations of motion in situations involving motion other than projectile motion or simple harmonic motion.

• Examine force, acceleration, action and reaction under constant and non-constant force.

• *Resisted motion*: Derive, from Newton's laws of motion, the equation of motion of a particle moving in a single direction under a resistance proportional to a power of the speed.

Newton's First law of motion

An object at rest tends to stay at rest. An object in motion tends to stay in motion in a straight line unless acted upon by an external force.

Newton's first law of motion, using vectors

Newton's first law states that

"Every object will remain at rest or in uniform motion in a straight line unless compelled to change its state by the action of an external force".

In terms of vectors, this statements is:

- If $\vec{a} = \vec{0}$ then
- *Velocity*: $\vec{v} =$ a constant vector $= \vec{v}_0$
- *Position*: $\vec{r}(t) = \vec{r}_0 + t\vec{v}_0$ where $\vec{r}_0$ and $\vec{v}_0$ are the position and velocity at $t = 0$.

Hey?!? Wait a minute, I have seen this before!

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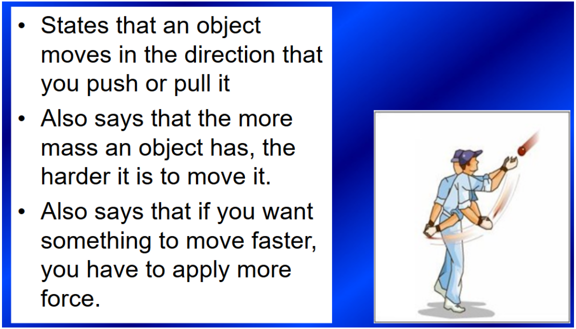
We recognise this as the equation of a line in parametric vector form! (This is the *"uniform motion in a straight line"* mentioned above.)

Newton's Second law of motion

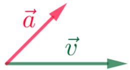
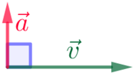
Second law: The vector sum of the forces $\vec{F}$ acting on an object is equal to the (constant) mass m of that object multiplied by the acceleration $\vec{a}$ of the object, i.e. $\sum \vec{F} = m\vec{a}$.



Is the object accelerating or slowing down?

Exercise 9: Tick the corret boxes.

Recall that the acceleration and the sum of the forces have the same direction since $\sum \vec{F} = m\vec{a}$. Thinking of the acceleration as more or less the same thing as the resulting force will help your intuition.

	accelerating	slowing down	constant velocity
			
			
			

Bonus: Which mathematical tool is hiding in the background?

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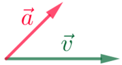
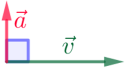
	accelerating	slowing down	constant velocity
		✓	
	✓		
			✓

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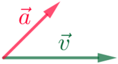
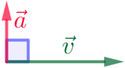
	✓		
			✓

Bonus: Which mathematical tool is hiding in the background?

The scalar (dot) product! Whether it is positive, negative or zero is the key here.

Newton's Laws of Motion in a nutshell

1. **First law**: If NO force is applied to an object (or if the forces compensate, i.e. if their sum is the zero vector), the object either remains at rest or continues to move at a constant velocity.

2. **Second law**: The sum of the forces $\vec{F}$ acting on an object is equal to the (constant) mass m of that object multiplied by the acceleration of the object:

$$\sum \vec{F} = m\vec{a}.$$

   N.B. The acceleration and the sum of the forces have the same direction

3. **Third law**: When one body exerts a force on a second body, the second body simultaneously exerts a force equal in magnitude and opposite in direction on the first body.

   E.g. Me, standing on the ground right now

PROJECTILE MOTION

Why study projectile motion?

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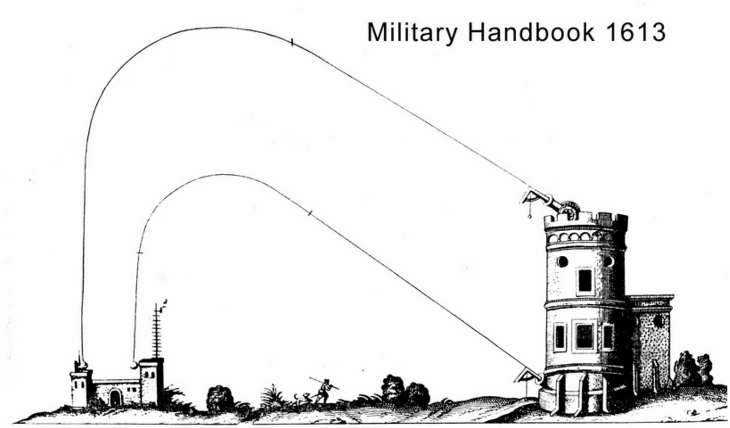

Projectile motion in the new syllabus [Extension 1]

Students:

• understand the concept of projectile motion, and model and analyse a projectile's path assuming that:

– the projectile is a point

– the force due to air resistance is negligible

– the only force acting on the projectile is the constant force due to gravity, assuming that the projectile is moving close to the Earth's surface.

• model the motion of a projectile as a particle moving with constant acceleration due to gravity and derive the equations of motion of a projectile

– represent the motion of a projectile using vectors

– recognise that the horizontal and vertical components of the motion of a projectile can be represented by horizontal and vertical vectors

– derive the horizontal and vertical equations of motion of a projectile

– understand and explain the limitations of this projectile model

• use equations for horizontal and vertical components of velocity and displacement to solve problems on projectiles

Projectile motion, the old way

For convenience we take the coordinates of the launch site to be $(x, y) = (0, 0)$.

- Start from the horizontal and vertical components of acceleration:
  $x''(t) = 0 \qquad y''(t) = -g$

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- Start from the horizontal and vertical components of acceleration:
  $x''(t) = 0 \qquad y''(t) = -g$
- Integrate to get the velocity:
  $x'(t) = v_0 \cos\theta \qquad y'(t) = -gt + v_0 \sin\theta$
  because, at $t = 0$,
  $v_{0,x} = x'(0) = v_0 \cos\theta$ and $v_{0,y} = y'(0) = v_0 \sin\theta$

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- Integrate to get the position:
  $x(t) = v_0 t \cos\theta \qquad y(t) = -\frac{1}{2} g t^2 + v_0 t \sin\theta$

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The new syllabus restates this using vectors.

Projectile motion: Visualisation using vectors

Look at [Geogebra Projectile Motion LHG] and fill in the blanks.

1. The components of the **acceleration** vector $\vec{a}$ are $\begin{pmatrix} \square \\ \square \end{pmatrix}$.
   The magnitude of $\vec{a}$ is $\square$.
2. The **velocity** vector $\vec{v}$ is always $\square$ to the trajectory (= the path of the moving object).
   The horizontal component of $\vec{v}$ is $\square$.
   At the highest point of the trajectory, $\vec{v}$ is $\square$
3. When the object is going $\square$, the angle between $\vec{a}$ and $\vec{v}$ is between 90° and 180° so the object is $\square$.
   However, when the object is going $\square$, the angle between $\vec{a}$ and $\vec{v}$ is between 0° and 90° so the object is $\square$.

Word Bank: up, down, horizontal, vertical, constant, tangent, slowing down, accelerating, g (standard gravity on Earth at sea level $\approx 9.81 m/s^2$)

Projectile motion: Visualisation using vectors

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2. The **velocity** vector $\vec{v}$ is always $\boxed{\text{tangent}}$ to the trajectory (= the path of the moving object).

   The horizontal component of $\vec{v}$ is $\boxed{\text{constant}}$.

   At the highest point of the trajectory, $\vec{v}$ is $\boxed{\text{horizontal}}$

Word Bank: up, down, horizontal, vertical, constant, tangent, slowing down, accelerating, g (standard gravity on Earth at sea level $\approx 9.81 m/s^2$)

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   The horizontal component of $\vec{v}$ is $\boxed{\text{constant}}$.
   At the highest point of the trajectory, $\vec{v}$ is $\boxed{\text{horizontal}}$
3. When the object is going $\boxed{\text{up}}$, the angle between $\vec{a}$ and $\vec{v}$ is between 90° and 180° so the object is $\boxed{\text{slowing down}}$.
   However, when the object is going $\boxed{\text{down}}$, the angle between $\vec{a}$ and $\vec{v}$ is between 0° and 90° so the object is $\boxed{\text{accelerating}}$.

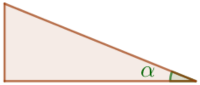
Word Bank: up, down, horizontal, vertical, constant, tangent, slowing down, accelerating, g (standard gravity on Earth at sea level $\approx 9.81 m/s^2$)

Benefits of using Vectors when studying Motion

Vectors and Newton's Laws of motion give an intuitive idea of what is going on (This is real life, not just calculations!).
They let you make predictions and check your work.

- You know which forces are applied to your particle: This tells you in which direction the acceleration vector is pointing (Thanks Isaac!).
  No force in a given direction? velocity should be constant in this direction.
- Velocity, as a vector, makes more sense than its two components separately: It is tangent to the trajectory and it points towards where you are going.
  If is is not tangent to the trajectory, check your work...
- Just by looking at $\vec{v}$ and $\vec{a}$, you can tell if the particle is accelerating or slowing down.

Example:
Going down a slope



Projectile Motion, revamped using vectors

For convenience we take the coordinates of the launch site to be $(x, y) = (0, 0)$.

- Use $m\vec{a} = m\vec{g}$ to get the acceleration:

$$\vec{a} = \vec{g} \text{ i.e. } \begin{pmatrix} x''(t) \\ y''(t) \end{pmatrix} = \begin{pmatrix} 0 \\ -g \end{pmatrix}$$

- Integrate to get the velocity vector:

$$\vec{v}(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} v_0 \cos\theta \\ -gt + v_0 \sin\theta \end{pmatrix}$$

because, at $t = 0$,

$$\overrightarrow{v_0} = \begin{pmatrix} v_0 \cos\theta \\ v_0 \sin\theta \end{pmatrix}.$$

- Integrate to get the position vector:

$$\overrightarrow{OR} = \vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} v_0 t \cos\theta \\ -\frac{1}{2} g t^2 + v_0 t \sin\theta \end{pmatrix}$$

Projectile motion - A Few Questions

We now have the model $\overrightarrow{OR} = \vec{r} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} v_0 t \cos\theta \\ -\frac{1}{2}gt^2 + v_0 t \sin\theta \end{pmatrix}$

Question 9: When does the projectile reach the peak of it's flight?

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Question 9: When does the projectile reach the peak of it's flight?

We can can answer this by looking for time when $y'(t) = 0$ (the velocity vector is tangent to the trajectory, so at the peak, it is horizontal).

Since $y'(t) = -gt + v_0 \sin\theta$, $y'(t) = 0$ has a unique solution, namely

$$t = \frac{v_0 \sin\theta}{g}.$$

Intuitively this should make sense are we are losing $g\,\mathrm{ms}^{-1}$ per second in the vertical direction, and we started with an initial $v_0 \sin\theta$ m s^{-1}.

Projectile motion - A Few Questions

Question 10:

We now have the model $\overrightarrow{OR} = \vec{r} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} v_0 t \cos\theta \\ -\frac{1}{2}gt^2 + v_0 t \sin\theta \end{pmatrix}$

What initial angle θ gives the greatest range?

Using Geogebra Projectile Motion, make a conjecture (i.e. an educated guess) and then prove it.

Projectile motion - A Few Questions

Question 10:

We now have the model $\overrightarrow{OR} = \vec{r} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} v_0 t \cos\theta \\ -\frac{1}{2}gt^2 + v_0 t \sin\theta \end{pmatrix}$

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We can can answer this by looking for time when $y(t) = 0$. Namely,

$$t_1 = 0 \qquad \text{or} \qquad t_2 = \frac{2v_0 \sin(\theta)}{g}.$$

Plug t_2 into the equation for x to get the range

$$x(t_2) = \frac{2v_0^2 \sin\theta \cos\theta}{g} = \frac{v_0^2 \sin 2\theta}{g}.$$

This will be maximal when $\sin 2\theta = 1$, and thus

$$\theta_{\text{optimal}} = \frac{\pi}{4}.$$

Projectile motion - A Few Questions

We now have the model $\overrightarrow{OR} = \vec{r} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} v_0 t \cos\theta \\ -\frac{1}{2}gt^2 + v_0 t \sin\theta \end{pmatrix}$

Question 11: How fast is the projectile going when it hits the ground?

Projectile motion - A Few Questions

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Question 11: How fast is the projectile going when it hits the ground?

We already know (see "greatest range" question) that the projectile will strike the ground at

$$t_2 = \frac{2v_0 \sin\theta}{g}.$$

Plugging this into our equations for the velocity vector gives

$$\vec{v}(t_2) = \begin{pmatrix} x'(t_2) \\ y'(t_2) \end{pmatrix} = \begin{pmatrix} v_0 \cos\theta \\ -v_0 \sin\theta \end{pmatrix} \quad \text{since} \quad \vec{v}(t) = \begin{pmatrix} v_0 \cos\theta \\ -gt + v_0 \sin\theta \end{pmatrix}$$

We can calculate the speed of the impact by calculating the magnitude of the velocity vector

$$|\vec{v}(t_2)| = \sqrt{v_0^2 \cos^2\theta + v_0^2 \sin^2\theta} = \sqrt{v_0^2} = |v_0| = v_0.$$

Limitations to the projectile model

Limitations to the projectile model

The model of projectile motion we explored ignored the effect of air-resistance. The effect of drag is typically proportional to the square of the velocity, so a more realistic model is given by

$$\begin{aligned} m\vec{a} = m\frac{d^2\vec{r}}{dt^2} &= m\vec{g} - k|\vec{v}|\vec{v} \\ &= m\vec{g} - k\left|\frac{d\vec{r}}{dt}\right|\frac{d\vec{r}}{dt}. \end{aligned}$$

where k is a parameter relating to the shape and size of an object.

$$\frac{d^2\vec{r}}{dt^2} = \vec{g} - \frac{k}{m}\left|\frac{d\vec{r}}{dt}\right|\frac{d\vec{r}}{dt}.$$

For arbitrary initial condition, this differential equation cannot be solved in terms of simple functions. However if the motion is restricted to only the y-axis (i.e. $\theta = 90^\circ$ and $x =$ constant) then this does have a closed form solution.

THANKS

... to your for coming,

... to Joshua Capel, UNSW, for his help.

BONUS MATERIAL

A Baccalaureat question ($\approx$ French HSC, Ext. 1)

EXERCISE 4 **Baccalauréat S ($\approx$ Ext. 1), Pondichéry 4 May 2018. 5 marks (Total = 20 marks in 4h)**

Consider the points $A(2\,,\,1\,,\,4)$, $B(4\,,\,-1\,,\,0)$, $C(0\,,\,3\,,\,2)$ and $D(4\,,\,3\,,\,-2)$.

1. Find a vector equation of the line CD.
2. Let M be a point on the line CD.
   a. Find the coordinates of the point M that make the length of BM minimal.
   b. Let H be the point on the line CD with coordinates $(3\,,\,3\,,\,-1)$. Verify that the line BH and CD are perpendicular.
   c. Show that the area of the triangle BCD is equal to 12 cm^2.
3. a. Show that the $\vec{n} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ is normal to the plane BCD.
   b. Determine a Cartesian equation of the plane through BCD.
   c. Find a vector equation of the line Δ through A and orthogonal to the plane through BCD.
   d. Prove that I, the point of intersection of the line Δ and the plane through BCD has coordinates $\left(\frac{2}{3}\,,\,\frac{1}{3}\,,\,\frac{8}{3}\right)$.
4. Calculate the volume of the tetrahedron $ABCD$.

Solutions **(in French, sorry. All Baccalaureate questions and answers are available in pdf and LaTeX.)**

Get some practice in 3D:

Question 12: Let ℓ_1 be the line defined by $\begin{cases} x + 3y - 2z = 4 \\ -3x + y + z = -2 \end{cases}$

1. Let ℓ_2 be the line defined by $\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ -3 \\ -6 \end{pmatrix}$, for $\lambda \in \mathbb{R}$.

   Are the lines ℓ_1 and ℓ_2 parallel?

2. Let Π be the plane with equation $-8x - 6y + 7z = 2018$.
   Is ℓ_1 parallel to that plane?

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   Are the lines ℓ_1 and ℓ_2 parallel?
2. Let Π be the plane with equation $-8x - 6y + 7z = 2018$.
   Is ℓ_1 parallel to that plane?

ANSWERS

1. Hint: Start by finding an equation for ℓ_1 in parametric vector form to get that $\vec{u_1} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ gives the direction of ℓ_1. Now, $\vec{u_2} = \begin{pmatrix} -3 \\ -3 \\ -6 \end{pmatrix}$, which gives the direction of ℓ_2, is a scalar multiple of $\vec{u_1}$. Therefore the lines ℓ_1 and ℓ_2 parallel.
2. Yes, calculate the scalar product of $\vec{u_1}$ with a vector normal to Π to prove it.